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THE CAUCHY INTEGRAL, CALDERÓN COMMUTATORS, AND CONJUGATIONS OF SINGULAR INTEGRALS IN R^n

BY

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ABSTRACT. We consider the Cauchy integral and Hilbert transform for Lipschitz domains in the Clifford algebra based on R^n. The Hilbert transform is shown to be the generating function for the Calderón commutators in R^n. We make use of an intrinsic characterization of these commutators to obtain L^2 estimates. These estimates are used to show the analyticity of the Hilbert transform and of the conjugation of singular integral operators by bi-Lipschitz changes of variable in R^n.

1. Introduction. In this paper, we study certain singular integral operators in R^n, via complex Clifford algebra. We show that some of the recent results in nonlinear harmonic analysis have natural higher-dimensional analogues in the setting of Clifford analysis. The primary advantage of Clifford analysis for our study is that estimates for certain nonlinear singular integral operators in R^n may now be obtained by an intrinsic characterization of these operators. Such estimates could previously be obtained only by using the "method of rotation" to reduce the problem to its one-dimensional analogue. In particular, the pioneering work of Calderón, Coifman, McIntosh, and Meyer [**3, 4**] on the L^2-boundedness of the Hilbert transform for Lipschitz curves in R^2 has a particularly remarkable extension to R^{n+1} in the framework of Clifford analysis.

Let us consider the classical formulation of this problem in R^2 (see [**2**]). Let A be a locally integrable, real-valued function which is the primitive of a function $a \in L^\infty(R)$, and suppose $f \in L^2(R)$. Let $\Gamma = \{x + iA(x)\colon x \in R\}$ be the Lipschitz curve in the complex plane given by the graph of A. The Cauchy integral of f on Γ is given by

$$(1.1) \qquad (Cf)(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(y)}{z-(y+iA(y))}(1+ia(y))\,dy$$

for $z \in C \setminus \Gamma$. It can be shown that the limit of $(Cf)(z)$, as z approaches Γ from above and nontangentially, is given by $\frac{1}{2}(I + iH_\Gamma)f(x)$, where I is the identity

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operator and H_Γ is the Hilbert transform for the Lipschitz curve, given by

$$(1.2)\qquad H_\Gamma f(x) = \frac{1}{\pi}\text{p.v.}\int_{-\infty}^{\infty} \frac{f(y)}{x - y + i(A(x) - A(y))}(1 + ia(y))\,dy.$$

It is easily seen that

$$(1.3)\qquad H_\Gamma f = \sum_{m=0}^{\infty} \frac{1}{m!}(-1)^m C_m(a, (1 + ia)f),$$

where, for $g \in L^2(R)$,

$$(1.4)\qquad C_m(a, g)(x) = i^m \frac{m!}{\pi}\text{p.v.}\int \left(\frac{A(x) - A(y)}{x - y}\right)^m \frac{g(y)}{x - y}\,dy.$$

C_m is the so-called Calderón commutator of order m; and in fact,

$$(1.5)\qquad C_m(a, g) = \delta_A^m(D^m H),$$

where $\delta_A(T) = [A, T]$ (here and in the sequel, the same symbol will be used to denote both a function and the operator of multiplication by that function), $D = -i(d/dx)$, and H is the Hilbert transform. Making use of the resolvent-type formula

$$(1.6)\qquad D^m H = -\frac{1}{\pi}\text{p.v.}\int_{-\infty}^{\infty} (tD)^m (I - itD)^{-1} \frac{dt}{t^{m+1}}$$

and the properties of the commutator δ_A, Coifman, McIntosh and Meyer were able to show **[4]** that $C_m(a, \cdot)$ has the integral representation

$$(1.7)\qquad -i^m \frac{m!}{\pi}\text{p.v.}\int_{-\infty}^{\infty} \{(P_t + iQ_t)a\}^m (P_t + iQ_t)\frac{dt}{t},$$

where $P_t = (I + t^2D^2)^{-1}$, $Q_t = tDP_t$. They were able to show that the integrand in (1.7) carries a sufficiently high degree of cancellation to assure the L^2 boundedness of the integral and, in turn, the summability of the series (1.3).

The analogues of the Calderón commutators in R^n are the operators of the form

$$(1.8)\qquad \delta_A^m(\Lambda^m R_j)\quad \text{for } m \text{ even},\qquad \delta_A^m(\Lambda^m)\quad \text{for } m \text{ odd},$$

where $A \in L^1_{\text{loc}}(R^n)$ has a bounded gradient, $\Lambda = (-\Delta)^{1/2}$ and R_j is the jth Riesz transform. Coifman, McIntosh and Meyer have shown that commutators of this type are bounded operators on $L^2(R^n)$; their proof (see **[4]**) is by reduction to the one-dimensional case.

In this paper we consider the commutators (1.8) from the perspective of function theory in a Clifford algebra. In this context the so-called regular (or monogenic) functions play the role of the analytic functions in complex function theory, and a Cauchy integral formula obtains for such functions. In §2 we define the Clifford algebra and establish basic notation. In §3 we consider the boundary behavior of Cauchy integrals for C^2 and minimally smooth domains, and define the Hilbert transform for Lipschitz hypersurfaces. In particular, we see that the Hilbert transform can be written as a series of Calderón commutators, analogous to (1.3). In §4, we obtain an integral representation formula for the commutators, analogous to

(1.7), which facilitates analysis of these operators. In §5, we use recent results of Coifman, Meyer and Stein on the Tent Spaces (see [**5**, **6**]) to prove the L^2 boundedness of the Hilbert transform. In §6, we consider applications of the L^2 estimates. In particular, we prove the analyticity of conjugation by a bi-Lipschitz change of variable for a broad class of Calderón-Zygmund operators.

2. Algebraic preliminaries. Let $A_n(C)$ denote the complex Clifford algebra based on C^n (see [**1** or **8**, Chapter 4]). $A_n(C)$ is a 2^n-dimensional algebra generated by the elements $e_0, e_1, \ldots, e_n$ subject to the relations

$$e_0 = 1, \tag{2.1}$$

$$e_j e_k + e_k e_j = -2\delta_{jk} \quad \text{for } 1 \le k,\, j \le n. \tag{2.2}$$

If $B = \{\beta_1, \ldots, \beta_s\}$ is a nonempty subset of $\{1, \ldots, n\}$ with $\beta_1 < \beta_2 < \cdots < \beta_s$, we write

$$e_B = e_{\beta_1} e_{\beta_2} \cdots e_{\beta_s} \tag{2.3}$$

and we set $e_\varnothing = e_0$. It is then easy to see that $\{e_B : B \subseteq \{1, \ldots, n\}\}$ is a basis for $A_n(C)$ as an algebra over C. If α is an arbitrary element of the algebra, we write

$$\alpha = \sum_B \alpha_B e_B, \tag{2.4}$$

where the summation is taken over all subsets B of $\{1, \ldots, n\}$ and each $\alpha_B \in C$. We define an involution of the algebra by setting

$$\bar{\alpha} = \sum_B \alpha_B \bar{e}_B, \tag{2.5}$$

where $\bar{e}_\varnothing = 1$, and for $B \neq \varnothing$ and card $B = s$,

$$\bar{e}_B = (-1)^{s(s+1)/2} e_B. \tag{2.6}$$

The spaces R^n and R^{n+1} (resp. C^n and C^{n+1}) will be identified with the subspaces spanned over R (resp. C) by $\{e_1, \ldots, e_n\}$ and $\{e_0, e_1, \ldots, e_n\}$, respectively. $A_n(R)$ denotes the subspace of elements having only real components; i.e., $\alpha = \sum_B \alpha_B e_B \in A_n(R)$ if and only if each α_B is real. If $\alpha \in A_n(R)$, we define $\operatorname{Re} \alpha = \alpha_\varnothing$.

We define an inner product on $A_n(C)$ by setting

$$(\alpha|\gamma) = \left(\sum_B \alpha_B e_B \middle| \sum_B \gamma_B e_B \right) = \sum_B \alpha_B \bar{\gamma}_B \tag{2.7}$$

which induces a norm

$$|\alpha|_0 = (\alpha|\alpha)^{1/2}. \tag{2.8}$$

Note that if $\alpha \in A_n(R)$, then $|\alpha|_0^2 = \operatorname{Re}(\alpha\bar{\alpha})$; if $\alpha \in R^{n+1}$, then $|\alpha|_0 = |\alpha|$, the ordinary Euclidean norm. We obtain an equivalent norm by defining

$$|\alpha|_x = \max_B |\alpha_B|. \tag{2.9}$$

We shall be concerned with the Hilbert space $L_0^2(R^n) = L^2(R^n, A_n(C))$ of algebra-valued, square-integrable functions on R^n, supplied with the inner product

$$(f|g) = \left(\sum_B f_B e_B \middle| \sum_B g_B e_B \right) = \sum_B \int_{R^n} f_B \bar{g}_B \tag{2.10}$$

and the induced norm

$$\|f\|_{0,2} = \left(\sum_B \|f_B\|_2^2 \right)^{1/2}. \tag{2.11}$$

We are also concerned with the Banach space $L_0^\infty(R^n) = L^\infty(R^n, A_n(C))$ of essentially-bounded algebra-valued functions on R^n, with the norm

$$\|f\|_{0,\infty} = \operatorname*{ess\,sup}_{x \in R^n} |f(x)|_0. \tag{2.12}$$

We have the equivalent norms

$$\|f\|_{x,2} = \max_B \|f_B\|_2, \tag{2.13}$$

$$\|f\|_{x,\infty} = \max_B \|f_B\|_\infty. \tag{2.14}$$

Note that $L^2(R^n, C)$ and $L^\infty(R^n, C)$ may be viewed as subspaces of $L_0^2(R^n)$ and $L_0^\infty(R^n)$ by associating each scalar function f to the algebra-valued function fe_0.

We introduce the following differential operators. If f is a differentiable, algebra-valued function on an open subset of R^{n+1}, we define

$$\mathscr{D}_0^L f = \sum_{j=0}^n \sum_B \frac{\partial f_B}{\partial x_j} e_j e_B, \tag{2.15}$$

$$\mathscr{D}_0^R f = \sum_{j=0}^n \sum_B \frac{\partial f_B}{\partial x_j} e_B e_j. \tag{2.16}$$

We shall sometimes write $\mathscr{D}_0 = \mathscr{D}_0^L$. Similarly, if g is a differentiable, algebra-valued function on an open subset of R^n, we define

$$\mathscr{D}^L g = \mathscr{D} g = \sum_{j=1}^n \sum_B \frac{\partial g_B}{\partial x_j} e_j e_B. \tag{2.17}$$

$\mathscr{D}_0^L$, $\mathscr{D}_0^R$ are analogues of the Cauchy-Riemann operator, while $\mathscr{D}$ is an embedding of the gradient in R^n into the algebra.

For $1 \le j \le n$, we set $D_j = -i\partial/\partial x_j$. We set $\Lambda^2 = -\Delta = \mathscr{D}^2$, so that the jth Riesz transform is given by $R_j = iD_j\Lambda^{-1}$. For $f \in \mathscr{S}(R^n)$, we define the Fourier transform according to the normalization

$$\hat{f}(\xi) = \int_{R^n} e^{-ix\cdot\xi} f(x)\, dx \tag{2.18}$$

so that we obtain

$$(D_j f)\hat{}(\xi) = \xi_j \hat{f}(\xi), \tag{2.19}$$

$$(\Lambda f)\hat{}(\xi) = |\xi| \hat{f}(\xi), \tag{2.20}$$

$$(R_j f)\hat{}(\xi) = i\xi_j |\xi|^{-1} \hat{f}(\xi). \tag{2.21}$$

If we set $\mathscr{R} = \mathscr{D}\Lambda^{-1}$, then for $f \in \mathscr{S}(R^n, A_n(C))$, we have

$$\mathscr{R} f = \sum_{j=1}^n \sum_B R_j f_B e_j e_B. \tag{2.22}$$

We observe that, if m is a nonnegative integer,

$$\mathscr{D}^m\mathscr{R} = \begin{cases} \Lambda^m\mathscr{R}, & m \text{ even}, \\ \Lambda^m, & m \text{ odd}. \end{cases} \tag{2.23}$$

3. Cauchy integrals. The main results of advanced calculus—the theorems of Green and Stokes—have natural extensions to the setting of Clifford analysis, and they serve as the underpinning for the development of function theory in Clifford algebras (see [**1** and **8**, Chapter 4]). The Clifford algebra analogue of analytic functions are the so-called *regular* (or *monogenic*) functions.

DEFINITION. Let $\Omega \subseteq R^{n+1}$ be open and let $f \in C^1(\Omega, A_n(C))$. We say that f is *left-regular* (resp. *right-regular*) in Ω if and only if $\mathscr{D}_0^L f = 0$ (resp. $\mathscr{D}_0^R f = 0$) in Ω.

Many of the results of analytic function theory have direct analogues in terms of regular functions. Of particular concern to us here is the Cauchy integral formula. We begin with some notation; set

$$dx = dx_0 \wedge dx_1 \wedge \cdots \wedge dx_n, \tag{3.1}$$

$$d\sigma(x) = \sum_{j=0}^{n} (-1)^j e_j d\hat{x}_j, \tag{3.2}$$

$$d\hat{x}_j = dx_0 \wedge dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_n. \tag{3.3}$$

For each $x \in R^{n+1}$, we define the function E_x on R^{n+1} by

$$E_x(u) = \frac{1}{\omega_{n+1}} \frac{\bar{u} - \bar{x}}{|u - x|^{n+1}}, \qquad u \in R^{n+1}, \tag{3.4}$$

where ω_{n+1} is the surface area of the unit sphere in R^{n+1}. We observe that E_x is both left- and right-regular in $R^{n+1} \setminus \{x\}$, and we have the following Cauchy integral formula.

THEOREM 3.1. *Suppose Ω is an open region in R^{n+1} and let $S \subseteq \Omega$ be an $(n+1)$-dimensional, compact, differentiable, oriented manifold-with-boundary. Suppose, moreover, that f is left-regular and g is right-regular in Ω. Then for each interior point x of S,*

$$f(x) = \int_{\partial S} E_x(u)\, d\sigma(u) f(u), \tag{3.5}$$

$$g(x) = \int_{\partial S} g(u)\, d\sigma(u) E_x(u). \tag{3.6}$$

For a proof, see [**1** or **8**].

Now let us consider the boundary behavior of Cauchy integrals on smooth domains. Henceforth we shall consider Cauchy integrals for left-regular functions; the case of right-regular functions is completely analogous. We first establish some notation. Let S be a bounded open subset of R^{n+1} such that the boundary, ∂S, is C^2, and such that $\partial S = \partial(R^{n+1} \setminus \bar{S})$, where $\bar{S}$ denotes the closure of S. For each point $x \in \partial S$, we let $\nu(x)$ denote the outer unit normal to ∂S at the point x. Now suppose

g is a continuous function on ∂S taking values in $A_n(C)$. If $x \in R^{n+1} \setminus \partial S$, we define

$$G(x) = \int_{\partial S} E_x(u)\, d\sigma(u) g(u). \tag{3.7}$$

We extend the kernel $E_x(u)$ to the boundary in the usual way. For $\varepsilon > 0$, let $B_\varepsilon(x)$ be the sphere of radius ε centered at x, and set $\Gamma_\varepsilon = \partial S \setminus (B_\varepsilon(x) \cap \partial S)$. Then, for $x \in \partial S$, we define

$$Kg(x) = \int_{\partial S} E_x(u)\, d\sigma(u) g(u) = \lim_{\varepsilon \to 0} \int_{\Gamma_\varepsilon} E_x(u)\, d\sigma(u) g(u); \tag{3.8}$$

i.e., the integral is taken in the principal value sense on the boundary. $G(x)$ is the Cauchy integral of the function g at the point x, and is evidently left-regular in $R^{n+1} \setminus \partial S$. We have the following result.

THEOREM 3.2. *Let g: $\partial S \to A_n(C)$ satisfy a Lipschitz condition of order α for some $\alpha \in (0,1]$, and let G be defined on $R^{n+1} \setminus \partial S$ by* (3.7). *Then*

$$\lim_{t \to 0-} G(x + t\nu(x)) = (\tfrac{1}{2}I + K)g(x), \tag{3.9}$$

$$\lim_{t \to 0+} G(x + t\nu(x)) = (-\tfrac{1}{2}I + K)g(x), \tag{3.10}$$

where K is defined by (3.8).

PROOF. The proof is entirely analogous to that of Theorem 3.22 of [7]. We begin by observing that

$$\int_{\partial S} E_x(u)\, d\sigma(u) = \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{if } x \in R^{n+1} \setminus \bar{S}, \\ 1/2 & \text{if } x \in \partial S, \end{cases} \tag{3.11}$$

which is a relatively straightforward consequence of the Cauchy integral formula and Cauchy's Theorem (cf. Corollary 9.3 of [1]; see also Proposition 3.19 of [7]). Now let $x \in \partial S$. If $|t|$ is sufficiently small, we have, by (3.11),

(3.12)

$$G(x + t\nu(x)) = \int_{\partial S} E_{x+t\nu(x)}(u)\, d\sigma(u)(g(u) - g(x)) + g(x) \quad \text{for } t < 0,$$

$$G(x + t\nu(x)) = \int_{\partial S} E_{x+t\nu(x)}(u)\, d\sigma(u)(g(u) - g(x)) \quad \text{for } t > 0. \tag{3.13}$$

By virtue of the smoothness assumption on g and (3.11), we have

$$\lim_{t \to 0} \int_{\partial S} E_{x+t\nu(x)}(u)\, d\sigma(u)(g(u) - g(x)) = Kg(x) - \tfrac{1}{2}g(x) \tag{3.14}$$

and the result follows. Q.E.D.

Next, let us consider Cauchy integrals defined on certain noncompact hypersurfaces in R^{n+1}. Specifically, suppose A: $R^n \to R$ satisfies a Lipschitz condition of order 1 (i.e., $a = \mathscr{D}A \in L^\infty(R^n, R^n)$) and let Γ denote its graph: $\Gamma = \{(A(x), x): x \in R^n\}$. The hypersurface Γ divides R^{n+1} into two regions:

(3.15) $$\Omega^+ = \{(x_0, x): x \in R^n, x_0 > A(x)\},$$

(3.16) $$\Omega^- = \{(x_0, x): x \in R^n, x_0 < A(x)\}.$$

To each function f_0 on Γ we may associate a function f on R^n as follows: define h: $R^{n+1} \to R^{n+1}$ by

(3.17) $$h(x_0, x) = (x_0 + A(x), x) \quad \text{for } x_0 \in R, x \in R^n.$$

Then define f to be the restriction of $f_0 \circ h$ to R^n, i.e. $f(x) = f_0(A(x), x)$. We then define the *Cauchy integral of f on* Γ by

(3.18) $$C_+f(x') = \int_\Gamma E_{x'}(u)\, d\sigma(u) f_0(u), \qquad x' \in \Omega^+,$$

(3.19) $$C_-f(x') = -\int_\Gamma E_{x'}(u)\, d\sigma(u) f_0(u), \qquad x' \in \Omega^-,$$

where, in each case, Γ is given the orientation induced by Ω^+. It is easy to see that

(3.20) $$C_+f(x') = \frac{1}{\omega_{n+1}} \int_{R^n} \frac{\bar{x}' - \bar{y} - A(y)}{|x' - y - A(y)|^{n+1}} (1 - a(y)) f(y)\, dy,$$

and an analogous formula holds for C_-f.

Suppose, for example, that $f \in C_0^\infty(R^n, A_n(C))$ and $A \in C^2(R^n, R)$. It is easy to see that C_+f (resp. C_-f) defines a left-regular function on Ω^+ (resp. Ω^-). Moreover, if $x \in R^n$ and $\eta(x)$ denotes the inner normal to Γ at $(x, A(x))$, then, by Theorem 3.2,

(3.21) $$\lim_{t \to 0+} C_+f(t\eta(x) + A(x), x) = \tfrac{1}{2} f(x) + \tfrac{1}{2} H_\Gamma f(x),$$

(3.22) $$\lim_{t \to 0-} C_-f(t\eta(x) + A(x), x) = \tfrac{1}{2} f(x) - \tfrac{1}{2} H_\Gamma f(x),$$

where

(3.23) $$H_\Gamma f(x) = \lim_{\varepsilon \to 0} H_{\Gamma,\varepsilon} f(x),$$

(3.24)
$$H_{\Gamma,\varepsilon} f(x) = \frac{2}{\omega_{n+1}} \int_{|x-y|>\varepsilon} \frac{\bar{x} - \bar{y} + A(x) - A(y)}{\left[|x-y|^2 + (A(x) - A(y))^2\right]^{(n+1)/2}} (1 - a(y)) f(y)\, dy.$$

$H_\Gamma f(x)$ is called the *Hilbert transform of f at x for the hypersurface* Γ, and $H_{\Gamma,\varepsilon}$ is called the *truncated Hilbert transform*.

In fact, we can relax the condition on the smoothness of A considerably. Suppose A is merely Lipschitz. Choose positive constants τ, ε for which the truncated cone

(3.25) $$V(x) = \{y' \in R^{n+1}: \tau > (y' - x - A(x)) \cdot \eta(x) > \varepsilon|y' - x - A(x)|\}$$

is contained entirely within Ω^+ for all $x \in R^n$. As in the dissertation of Verchota **[12]** it can be seen without difficulty that the nontangential limit of C_+f,

(3.26) $$\lim_{y' \to (A(x), x)} \text{n.t. } C_+f(y') = \lim_{\substack{y \in V(x) \\ y' \to (A(x), x)}} C_+f(y'),$$

when it exists, is equal to $\frac{1}{2} f(x) + \frac{1}{2} H_\Gamma f(x)$ almost everywhere.

A necessary and sufficient condition, therefore, for f to be the boundary value of its Cauchy integral in Ω^+ (resp. Ω^-) is that $f = H_\Gamma f$ (resp. $f = -H_\Gamma f$).

It is not at all difficult to see that

$$H_\Gamma f = \sum_{m=0}^{\infty} \frac{1}{m!} C_m(a, (1-a)f), \tag{3.27}$$

where

$$C_m(a, g)(x) = \begin{cases} \gamma(m)\text{p.v.}\int \left(\dfrac{A(x)-A(y)}{|x-y|}\right)^m \dfrac{\bar{x}-\bar{y}}{|x-y|^{n+1}} g(y)\,dy, & m \text{ even}, \\ \gamma(m)\text{p.v.}\int \left(\dfrac{A(x)-A(y)}{|x-y|}\right)^m \dfrac{1}{|x-y|^{n}} g(y)\,dy, & m \text{ odd} \end{cases} \tag{3.28}$$

and

$$\gamma(m) = \frac{(-1)^k \Gamma(k+(n+1)/2)}{k!\pi^{(n+1)/2}}, \tag{3.29}$$

where $k = [m/2]$, the greatest integer function at $m/2$. An elementary but tedious computation (see [**10**, Chapter 3]) shows that

$$C_m(a, \cdot) = \delta_A^m(\mathscr{D}^m \mathscr{R}), \tag{3.30}$$

where, for an operator T, $\delta_A(T) = [A, T]$, and, by abuse of notation, A is allowed to signify the operator of pointwise multiplication by A. C_m shall be called the *Calderón commutator of order m* for R^n. As in the one-dimensional case, we have see that the Hilbert transform for Γ is the generating function for these commutators.

4. Representation formulas for the Hilbert transform and Calderón commutators. We shall obtain L^2 estimates for the Calderón commutators in terms of the L^2 norm of f and the L^∞ norm of a; it suffices to establish these estimates under the assumption that both f and A are infinitely differentiable functions with compact support. We shall first obtain integral representations for the commutators, and then apply the methods of multilinear analysis to obtain L^2 estimates.

Suppose that f_0 is the boundary value of a function U_0 which is left-regular in Ω^+, and set $U = U_0 \circ h$, $f = f_0 \circ h$, where h is defined by (3.17). Let V_h denote the operator of composition by h, and V_h^{-1} its inverse. We must have

$$V_h \mathscr{D}_0 V_h^{-1} U = 0, \qquad \lim_{x_0 \to 0+} U(x_0, x) = f(x). \tag{4.1}$$

An elementary computation shows that

$$V_h \mathscr{D}_0 V_h^{-1} U = \mathscr{D} U + (1-a)\frac{\partial U}{\partial x_0} \tag{4.2}$$

so that we must have

$$\frac{\partial U}{\partial x_0} = -\frac{1}{1-a}\mathscr{D} U, \qquad \lim_{x_0 \to 0+} U(x_0, x) = f(x). \tag{4.3}$$

The problem is solved by setting

$$U(x_0, x) = \exp\left[-x_0(1-a)^{-1}\mathscr{D}\right] f(x); \tag{4.4}$$

proceeding formally, we have

$$\exp\left[-x_0(1-a)^{-1}\mathscr{D}\right] f = \frac{1}{\pi i}\text{p.v.}\int_{-\infty}^{\infty} e^{ix_0/t}\left(I - it(1-a)^{-1}\mathscr{D}\right)^{-1} f\,\frac{dt}{t}. \tag{4.5}$$

As $x_0 \to 0+$, we obtain

$$f = \frac{1}{\pi i}\text{p.v.}\int_{-\infty}^{\infty}\left(I - it(1-a)^{-1}\mathscr{D}\right)^{-1} f\,\frac{dt}{t}. \tag{4.6}$$

But we have already seen that f is the boundary value of its Cauchy integral if and only if $f = H_\Gamma f$. We claim that, in fact, if f is *any* function in $C_0^\infty(R^n, A_n(C))$, we may write

$$H_\Gamma f = \frac{1}{\pi i}\text{p.v.}\int_{-\infty}^{\infty}\left(I - it(1-a)^{-1}\mathscr{D}\right)^{-1} f\,\frac{dt}{t}. \tag{4.7}$$

It is easily seen that the right-hand side of (4.7) is equal to

$$\sum_{m=0}^{\infty}\frac{1}{\pi i}\text{p.v.}\int_{-\infty}^{\infty}\left[(I - it\mathscr{D})^{-1}a\right]^m (I - it\mathscr{D})^{-1}(1-a) f\,\frac{dt}{t} \tag{4.8}$$

so that, in view of (3.27) and (3.30), our claim holds if and only if for every $A \in C_0^\infty(R^n, R)$ and every $g \in C_0^\infty(R^n, A_n(C))$,

$$\delta_A^m(\mathscr{D}^m\mathscr{R})g = \frac{m!}{\pi i}\text{p.v.}\int_{-\infty}^{\infty}\left[(I - it\mathscr{D})^{-1}a\right]^m (I - it\mathscr{D})^{-1} g\,\frac{dt}{t}. \tag{4.9}$$

In fact, we will establish (4.9) for complex-valued A.

If we set $P_t = (I + t^2\Lambda^2)^{-1}$ and $Q_t = t\mathscr{D}P_t$, we obtain $(I \pm it\mathscr{D})^{-1} = P_t \mp iQ_t$, so that (4.9) becomes

$$\delta_A^m(\mathscr{D}^m\mathscr{R})g = \frac{m!}{\pi i}\text{p.v.}\int_{-\infty}^{\infty}\left[(P_t + iQ_t)a\right]^m (P_t + iQ_t) g\,\frac{dt}{t}. \tag{4.10}$$

To prove the equality (4.10), we begin by observing that

$$\mathscr{R} = \frac{1}{\pi i}\text{p.v.}\int_{-\infty}^{\infty}(P_t + iQ_t)\,\frac{dt}{t} = \frac{1}{\pi i}\int_0^{\infty}\{(P_t + iQ_t) - (P_t - iQ_t)\}\,\frac{dt}{t}, \tag{4.11}$$

so that

$$\begin{aligned}\mathscr{D}^m\mathscr{R} &= \frac{1}{\pi i}\text{p.v.}\int_{-\infty}^{\infty}(t\mathscr{D})^m(P_t + iQ_t)\,\frac{dt}{t^{m+1}} \\ &= \frac{1}{\pi i}\int_0^{\infty}\{(t\mathscr{D})^m(P_t + iQ_t) - (t\mathscr{D})^m(P_t - iQ_t)\}\,\frac{dt}{t^{m+1}}.\end{aligned} \tag{4.12}$$

Consequently,

$$\begin{aligned}\delta_A^m(\mathscr{D}^m\mathscr{R}) &= \frac{1}{\pi i}\text{p.v.}\int_{-\infty}^{\infty}\delta_A^m\left((t\mathscr{D})^m(P_t + iQ_t)\right)\frac{dt}{t^{m+1}} \\ &= \frac{1}{\pi i}\int_0^{\infty}\delta_A^m\left((t\mathscr{D})^m(P_t + iQ_t) - (t\mathscr{D})^m(P_t - iQ_t)\right)\frac{dt}{t^{m+1}}.\end{aligned} \tag{4.13}$$

It is not difficult to show that δ_A is a derivation of the algebra of continuous linear operators on the Schwartz class $\mathscr{S}(R^n, A_n(C))$. Moreover, all of the properties of δ_A as a derivation of the algebra of continuous linear operators on $\mathscr{S}(R, C)$, enumerated in **[4]**, have analogues in the Clifford algebra setting. In particular, it is easily seen that

$$\delta_A^m\big((t\mathscr{D})^m(P_t + iQ_t)\big) = t^m m!\{(P_t + iQ_t)a\}^m(P_t + iQ_t), \tag{4.14}$$

$$\delta_A^m\big((t\mathscr{D})^m(P_t - iQ_t)\big) = t^m m!\{(P_t - iQ_t)a\}^m(P_t - iQ_t), \tag{4.15}$$

whence we obtain

THEOREM 4.1. *If* $g \in C_0^\infty(R^n, A_n(C))$, $A \in C_0^\infty(R^n, C)$, *and* $a = \mathscr{D}A$, *then*

$$\begin{aligned} C_m(a, g) &= \frac{m!}{\pi i} p.v.\int_{-\infty}^{\infty} [(P_t + iQ_t)a]^m(P_t + iQ_t)g\,\frac{dt}{t} \\ &= \frac{m!}{\pi i}\int_0^{\infty} \{[(P_t + iQ_t)a]^m(P_t + iQ_t) \\ &\qquad - [(P_t - iQ_t)a]^m(P_t - iQ_t)\}g\,\frac{dt}{t}. \end{aligned} \tag{4.16}$$

Moreover, if $A \in C_0^\infty(R^n, R)$ *and* $f \in C_0^\infty(R^n, A_n(C))$,

$$H_\Gamma f = \frac{1}{\pi i} p.v.\int_{-\infty}^{\infty} \left(I - it(1 - a)^{-1}\mathscr{D}\right)^{-1} f\,\frac{dt}{t}. \tag{4.17}$$

Following **[4]**, we may write

$$C_m(a, \cdot) = \frac{m!}{\pi i}\left\{\sum_{\substack{p+q+r=m-1\\ p,q,r\ge 0}} \left(\mathscr{L}^-_{p,q,r} - \mathscr{L}^+_{p,q,r}\right) + 2i\sum_{\substack{p+q=m\\ p,q\ge 0}} \mathscr{L}_{p,q}\right\}, \tag{4.18}$$

where

$$\mathscr{L}^-_{p,q,r} = \int_0^\infty (P_t a)^p\{a(P_t - iQ_t)\}^q aQ_t(aP_t)^r\,\frac{dt}{t}, \tag{4.19}$$

$$\mathscr{L}^+_{p,q,r} = \int_0^\infty (P_t a)^p\{a(P_t + iQ_t)\}^q aQ_t(aP_t)^r\,\frac{dt}{t}, \tag{4.20}$$

$$\mathscr{L}_{p,q} = \int_0^\infty (P_t a)^p Q_t(aP_t)^q\,\frac{dt}{t} \tag{4.21}$$

for nonnegative integers p, q, r. In other words, $C_m(a, \cdot)$ may be written as a constant times the sum of $m(m + 1)$ operators of the form $\mathscr{L}^\pm_{p,q,r}$ and $2(m + 1)$ operators of the form $\mathscr{L}_{p,q}$. Thus analysis of the commutators is reduced to the analysis of the operators (4.19)–(4.21).

5. Reduction to quadratic estimates. For each positive integer j, $1 \le j \le n$, we define the scalar operator $Q_{j,t}$ by setting $Q_{j,t} = tD_jP_t$, so that

$$Q_t = i\sum_{j=1}^n Q_{j,t}e_j. \tag{5.1}$$

We shall sometimes write $P_1 = P$ and $Q_j = D_jP$. For $1 \le j \le n$, we define $Y_j = Q_j$, and we set $Y_{n+1} = P$. If $A \in C_0^\infty(R^n, C)$, then

$$a = \mathscr{D}A = \sum_{j=1}^{n} a_j e_j, \quad \text{where } a_j = \frac{\partial A}{\partial x_j}. \tag{5.2}$$

Now suppose that m is a positive integer, and let p, q, r be nonnegative integers with $p + q + r = m - 1$. Let $\alpha = (\alpha_1, \dots, \alpha_m)$ be a multi-index of positive integers between 1 and n, and let $\gamma = (\gamma_{p+2}, \gamma_{p+3}, \dots, \gamma_{p+q+1})$ be a multi-index of positive integers between 1 and $n + 1$. For $p, q, r \ge 1$, we define the operators

$$M_1(t) = M_1(\alpha, p, t) = \prod_{j=1}^{p} \left(P_t a_{\alpha_j}\right), \tag{5.3}$$

$$M_2(t) = M_2(\alpha, \gamma, q, t) = \prod_{j=p+2}^{p+q+1} \left(Y_{\gamma_j, t} a_{\alpha_j}\right), \tag{5.4}$$

$$M_3(t) = M_3(\alpha, r, t) = \prod_{j=p+q+2}^{m} \left(a_{\alpha_j} P_t\right), \tag{5.5}$$

and, for $p, q, r = 0$ we set $M_1(t) = M_2(t) = M_3(t) = I$, the identity operator. Now consider the scalar operator

$$\begin{aligned} L &= L(\alpha, \nu, \gamma, p, q, r) \\ &= \int_0^\infty M_1(t) Q_{\nu_1, t} a_{\alpha_{p+1}} M_2(t) Q_{\nu_2, t} M_3(t) \frac{dt}{t}. \end{aligned} \tag{5.6}$$

If we let

$$L^*(a) = \sup \| L(\alpha, \nu, \gamma, p, q, r) \|_{\text{op}} \tag{5.7}$$

(where the supremum is taken over all possible choices of α, ν, γ, p, q, r and $\|\cdot\|_{\text{op}}$ is the norm as an operator on $L^2(R^n, C)$), then there is a dimensional constant $C(n)$ such that

$$\left\| \sum_{\substack{p+q+r=m-1 \\ p,q,r \ge 0}} \left(\mathscr{L}^-_{p,q,r} - \mathscr{L}^+_{p,q,r}\right) \right\|_{\text{op}} \le (C(n))^m L^*(a) \tag{5.8}$$

(where the operator norm is here taken as that for operators on $L^2(R^n, A_n(C))$).

We can calculate the operator norm of L by duality. Let $f, g \in L^2(R^n, C)$, with $\|f\|_2 = \|g\|_2 = 1$. Then

$$\begin{aligned} &\left| \int_{R^n} g(x) Lf(x)\, dx \right| \\ &= \left| \int \int_{R_+^{n+1}} g(x) \left\{ M_1(t) Q_{\nu_1, t} a_{\alpha_{p+1}} M_2(t) Q_{\nu_2, t} M_3(t) f \right\}(x) \frac{dx\, dt}{t} \right| \\ &= \left| \int \int_{R_+^{n+1}} \left\{ Q_{\nu_1, t} \widetilde{M_1(t)} g \right\}(x) \left\{ a_{\alpha_{p+1}} M_2(t) Q_{\nu_2, t} M_3(t) f \right\}(x) \frac{dx\, dt}{t} \right|, \end{aligned} \tag{5.9}$$

where $\widetilde{M_1(t)} = \widetilde{M_1(\alpha, p, t)}$ is equal to I for $p = 0$, and

$$\widetilde{M_1(t)} = \prod_{j=0}^{p-1} (\alpha_{p-j} P_t) \quad \text{if } p \geq 1. \tag{5.10}$$

If we let $\|\cdot\|_2^+$ denote the norm on $L^2(R_+^{n+1}, dx dt/t)$, i.e.

$$\|F\|_2^+ = \left(\iint_{R_+^{n+1}} |F(x,t)|^2 \frac{dx\,dt}{t} \right)^{1/2}, \tag{5.11}$$

then we have

$$\left| \int_{R^n} g(x) Lf(x)\, dx \right| \leq \left\| Q_{\nu_1,t} \widetilde{M_1(t)} g \right\|_2^+ \left\| a_{\alpha_{p+1}} M_2(t) Q_{\nu_2,t} M_3(t) f \right\|_2^+ \tag{5.12}$$

by the Schwarz inequality. It is easy to see that $a_{\alpha_{p+1}} M_2(t)$ is a bounded operator on $L^2(R^n, C)$; in fact

$$\left\| a_{\alpha_{p+1}} M_2(t) \right\|_{\text{op}} \leq \|a\|_{x,\infty}^{q+1} \tag{5.13}$$

since $\|Y_{j,t}\|_{\text{op}} \leq 1$. Thus we have

$$\|L\|_{\text{op}} \leq \|a\|_{x,\infty}^{q+1} \sup_{\|f\|_2 = 1} \left\| Q_{\nu_2,t} M_3(t) f \right\|_2^+ \sup_{\|g\|_2 = 1} \left\| Q_{\nu_1,t} \widetilde{M_1(t)} g \right\|_2^+. \tag{5.14}$$

Thus the problem of estimating the operator norm of L (and consequently of $\mathscr{L}_{p,q,r}^+$ and $\mathscr{L}_{p,q,r}^-$) is reduced to that of estimating quadratic expressions of the form

$$\left\| Q_{j,t} \prod_{l=1}^{k} (a_{\alpha_l} P_t) f \right\|_2^+. \tag{5.15}$$

Similarly, analysis of $\mathscr{L}_{p,q}$ can be reduced to the same quadratic estimates. As in the one-dimensional case, the operators P_t and Q_t satisfy certain identities which greatly facilitate the analysis. We have

PROPOSITION 5.1. *Let P_t, Q_t be as above, and set $A_t = (2P_t - I)Q_t$. Then*

$$P_t = P_t^2 + Q_t^2, \tag{5.16}$$

$$t \frac{\partial}{\partial t} P_t = -2Q_t^2, \tag{5.17}$$

$$t \frac{\partial}{\partial t} Q_t = A_t, \tag{5.18}$$

$$t \frac{\partial}{\partial t} A_t = Q_t - 8Q_t^3. \tag{5.19}$$

PROOF. The proof is straightforward, making use of the fact that P_t, Q_t, $\mathscr{D}$ and Λ all commute (see also [4, Proposition 2]). Q.E.D.

By virtue of (5.19), we see that

$$\mathscr{L}_{p,q} = \mathscr{S}_{p,q} + 8\mathscr{V}_{p,q}, \tag{5.20}$$

where

$$\mathscr{S}_{p,q} = \int_0^\infty (P_t a)^p t \frac{\partial}{\partial t} A_t (a P_t)^q \frac{dt}{t} \tag{5.21}$$

and

$$\mathscr{V}_{p,q} = \int_0^\infty (P_t a)^p Q_t^3 (aP_t)^q \frac{dt}{t}. \tag{5.22}$$

Integration by parts, together with (5.17), shows that

$$\mathscr{S}_{p,q} = 2\sum_{j=1}^{p} \mathscr{S}_{p,j;q} + 2\sum_{j=1}^{q} \mathscr{S}_{p;q,j}, \tag{5.23}$$

where, for $q \geq 0$ and $1 \leq j \leq p$,

$$\mathscr{S}_{p,j;q} = \int_0^\infty (P_t a)^{j-1} Q_t^2 (aP_t)^{p-j} aA_t (aP_t)^q \frac{dt}{t} \tag{5.24}$$

and

$$\mathscr{S}_{p;q,j} = \int_0^\infty (P_t a)^p A_t a (P_t a)^{j-1} Q_t^2 (aP_t)^{q-j} \frac{dt}{t}. \tag{5.25}$$

As before, the analysis of $\mathscr{V}_{p,q}$, $\mathscr{S}_{p,j;q}$ and $\mathscr{S}_{p;q,j}$ proceeds by decomposing each operator into its scalar components and estimating the norms of the scalar operators by duality. We are thereby reduced yet again to estimating quadratic expressions of the form (5.15).

Let k be a positive integer, $\alpha = (\alpha_1, \ldots, \alpha_k)$ a multi-index of positive integers between 1 and n. We define

$$M(k,t) = M(\alpha, k, t) = \prod_{l=1}^{k} (a_{\alpha_l} P_t) \tag{5.26}$$

and set $M(0,t) = I$. We shall show that there is a dimensional constant $K(n)$ for which

$$\|Q_{j,t} M(k,t) f\|_2^+ \leq (K(n))^k \|a\|_{x,\infty}^k \|f\|_2 \tag{5.27}$$

for all $j \in \{1, \ldots, n\}$, k and α as above, $a \in L^\infty(R^n, C^n)$ and $f \in L^2(R^n, C)$. In view of (5.8), (5.14) and the analogous results for the operators $\mathscr{L}_{p,q}$, we shall then conclude that

$$\|C_m(a,f)\|_{0,2} \leq m!(C(n))^m \|a\|_{x,\infty}^m \|f\|_{0,2}, \tag{5.28}$$

where $C(n)$ is a purely dimensional constant. We begin with

LEMMA 5.2. *There is a dimensional constant $K(n) > 0$ such that if $1 \leq j \leq n$ and $g(x,t) = g_t(x)$ and $f(x,t) = f_t(x)$ are two functions on R_+^{n+1} satisfying*

$$\gamma = \sup_{t>0} |g_t| \quad \textit{is an element of } L^2(R^n), \tag{5.29}$$

$$f_t(x) \quad \textit{is an element of } L^2\left(R_+^{n+1}, \frac{dx\,dt}{t}\right), \tag{5.30}$$

then

$$G_j = D_j \int_0^\infty (P_t f_t)(P_t g_t)\,dt \quad \textit{is an element of } H^1(R^n), \tag{5.31}$$

the atomic Hardy space, and

$$\|G_j\|_{H^1} \le K(n)\|f_t\|_2^+ \|\gamma\|_2. \tag{5.32}$$

PROOF. The operator P_t is given by convolution with an L^1 function $p_t(x) = t^{-n}p(xt^{-1})$, and we may write

$$p(x) = \sum_{k=0}^{\infty} 4^{-k} p_k(x), \tag{5.33}$$

where $\hat{p}_k$ is supported in the ball of radius 2^k centered at 0; and moreover, there is a constant C independent of k such that

$$|D^\beta \hat{p}_k(\xi)| \le C2^{-|\beta|k} \tag{5.34}$$

for all multi-indices $\beta = (\beta_1, \ldots, \beta_n)$ such that $|\beta| = \beta_1 + \cdots + \beta_n \le n + 1$. Consequently, we may write

$$G_j = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} 4^{-k} 4^{-l} G(j; k, l), \tag{5.35}$$

where

$$G(j; k, l) = D_j \int_0^\infty (p_{k,t} * f_t)(p_{l,t} * g_t)\, dt. \tag{5.36}$$

As a consequence of recent work of Coifman, Meyer and Stein on the theory of Tent Spaces [**5**, **6**] we see that $G(j; k, l)$ is in $H^1(R^n)$, and moreover,

$$\|G(j; k, l)\|_{H^1} \le K_0(n) C^2 (2^k + 2^l) \|f_t\|_2^+ \|\gamma\|_2 \tag{5.37}$$

for a purely dimensional constant $K_0(n)$. Therefore

$$\begin{aligned} \|G_j\|_{H^1} &\le K_0(n) C^2 \|f_t\|_2^+ \|\gamma\|_2 \sum_k \sum_l (2^k + 2^l) 4^{-k} 4^{-l} \\ &\le K(n) \|f_t\|_2^+ \|\gamma\|_2 \end{aligned} \tag{5.38}$$

for a purely dimensional constant $K(n)$. Q.E.D.

LEMMA 5.3. *There is a dimensional constant $K(n) > 0$ such that if $b \in L^\infty(R^n)$, g_t and γ are as in Lemma 5.2 and $1 \le j \le n$, then*

$$\|Q_{j,t} b P_t g_t\|_2^+ \le \left(K(n)\|\gamma\|_2 + \|Q_{j,t} g_t\|_2^+\right)\|b\|_\infty. \tag{5.39}$$

PROOF. It is easily seen that

$$Q_{j,t} b P_t g_t = P_t\left([tD_j b] \cdot [P_t g_t]\right) + P_t b Q_{j,t} g_t \tag{5.40}$$

whence

$$\begin{aligned} \|Q_{j,t} b P_t g_t\|_2^+ &\le \left\|P_t\left([tD_j b] \cdot [P_t g_t]\right)\right\|_2^+ + \|P_t b Q_{j,t} g_t\|_2^+ \\ &\le \left\|P_t\left([tD_j b] \cdot [P_t g_t]\right)\right\|_2^+ + \|b\|_\infty \|Q_{j,t} g_t\|_2^+. \end{aligned} \tag{5.41}$$

Now

$$\left\| P_t\left(\left[tD_jb\right]\cdot\left[P_tg_t\right]\right)\right\|_2^+ \tag{5.42}$$
$$= \sup_{\|f_t\|_2^+=1}\left|\int\int_{R_+^{n+1}} f_t(x)P_t\left(\left[tD_jb\right]\cdot\left[P_tg_t\right]\right)(x)\,\frac{dx\,dt}{t}\right|$$
$$= \sup_{\|f_t\|_2^+=1}\left|\int_{R^n} b(x)\left(D_j\int_0^\infty \left[P_tf_t\right](x)\left[P_tg_t\right](x)\,dt\right)dx\right|$$

which is dominated by $K(n)\|\gamma\|_2\|b\|_\infty$, by Lemma 5.2 and the duality of $H^1(R^n)$ and BMO(R^n). Q.E.D.

LEMMA 5.4. *There is a dimensional constant $K(n)$ such that*

$$\left\|Q_{j,t}M(k,t)f\right\|_2^+ \le (K(n))^k\|a\|_{x,\infty}^k\|f\|_2. \tag{5.43}$$

PROOF. If $k=0$, we have $Q_{j,t}M(0,t)f = Q_{j,t}f$ and

$$\left\|Q_{j,t}f\right\|_2^+ = \left(\int_0^\infty\int_{R^n}\left|Q_{j,t}f(x)\right|^2dx\,\frac{dt}{t}\right)^{1/2} \tag{5.44}$$
$$= (2\pi)^{-n/2}\left(\int_{R^n}|\hat{f}(\xi)|^2\int_{-\infty}^\infty t^2\xi_j^2\left(1+t^2|\xi|^2\right)^{-2}\frac{dt}{t}\,d\xi\right)^{1/2}$$
$$\le (2\pi)^{-n/2}\left(\frac{1}{2}\int_{R^n}|\hat{f}(\xi)|^2\,d\xi\right)^{1/2} = \frac{1}{\sqrt{2}}\|f\|_2.$$

If $k\ge 1$, we have

$$Q_{j,t}M(k,t)f = Q_{j,t}a_{\alpha_1}P_tM(k-1,t)f. \tag{5.45}$$

Now let $g_{k-1,t}=M(k-1,t)f$, and set

$$\gamma_{k-1} = \sup_{t>0}|g_{k-1,t}|. \tag{5.46}$$

We claim that there is a constant $C(n)$ such that

$$\|\gamma_{k-1}\|_2 \le (C(n))^{k-1}\|a\|_{x,\infty}^k\|f\|_2. \tag{5.47}$$

For $k=1$, this is obvious. If $k>1$,

$$|g_{k-1,t}| \le |a_{\alpha_2}||P_t||a_{\alpha_3}||P_t|\cdots|a_{\alpha_k}||P_t||f| \le \|a\|_{x,\infty}^{k-1}|P_t|^{k-1}|f|, \tag{5.48}$$

where $|P_t|$ is the operator of convolution with the function $|p_t|$. Since $|p|$ decays rapidly at infinity, we see that

$$\sup_{t>0}|P_t|^{k-1}|f| \le M^{k-1}f, \tag{5.49}$$

where Mf is the Hardy-Littlewood maximal function of f (see [**10**, Chapter 3]). Since M is a bounded operator on L^2, we have (5.47). Combining (5.45), (5.47) and Lemma 5.3, the result follows. Q.E.D.

Thus we have proven

THEOREM 5.5. *There is a purely dimensional constant $C(n)$ such that*

$$\|C_m(a,f)\|_{0,2} \le m!(C(n))^m \|a\|_{x,\infty}^m \|f\|_{0,2}. \tag{5.50}$$

By a slight modification of our arguments, we obtain the following more general result.

THEOREM 5.6. *Let $A^1, A^2, \dots, A^m \in \mathrm{Lip}_1(R^n, C)$, and define, for $1 \le l \le n$,*

$$K_{l,m} = \begin{cases} \Lambda^m R_l, & m \text{ even}, \\ \Lambda^m, & m \text{ odd}, \end{cases} \tag{5.51}$$

$$T_{l,m}(A^1,\dots,A^m) = \delta_{A^1} \circ \delta_{A^2} \circ \cdots \circ \delta_{A^m}(K_{l,m}). \tag{5.52}$$

Then there is a purely dimensional constant $C(n)$ such that

$$\|T_{l,m}(A^1,\dots,A^m)f\|_2 \le m!(C(n))^m \left(\prod_{j=1}^m \|\nabla A^j\|_\infty\right) \|f\|_2 \tag{5.53}$$

for every $f \in L^2(R^n, C)$.

We have established

THEOREM 5.7. *There is a purely dimensional constant $C(n)$ such that for all $A \in \mathrm{Lip}_1(R^n, C)$ with $\|a\|_{x,\infty} < C(n)^{-1}$ and for all $g \in L^2(R^n, A_n(C))$,*

$$\left\| \sum_{m=0}^\infty \frac{1}{m!} C_m(a,g) \right\|_{0,2} \le \left(1 - C(n)\|a\|_{x,\infty}\right)^{-1} \|g\|_{0,2}. \tag{5.54}$$

In particular, for all $A \in \mathrm{Lip}_1(R^n, R)$ with $\|a\|_{x,\infty} < C(n)^{-1}$ and all $f \in L^2(R^n, A_n(C))$,

$$\|H_\Gamma f\|_{0,2} \le \left(1 - C(n)\|a\|_{x,\infty}\right)^{-1} (1 + \|a\|_{x,\infty}) \|f\|_{0,2}. \tag{5.55}$$

In other words, H_Γ is the restriction to real-valued functions A of a mapping, from $\mathrm{Lip}_1(R^n, C)$ to the space of bounded operators on $L^2(R^n, A_n(C))$, which is complex-analytic in a neighborhood of the origin.

6. Conjugations of singular integrals by changes of variable in R^n. Let A be a real-valued function on R, which is the primitive of a function a in the unit ball of $L^\infty(R)$, and set $h(x) = x + A(x)$. We obtain a bi-Lipschitz change of variable in R when we define U_h by $U_h f = f \circ h$. Then, if we conjugate the Hilbert transform by this change of variable, we obtain

$$\begin{aligned} U_h H U_h^{-1} &= \frac{1}{\pi} \mathrm{p.v.} \int \frac{1}{x - y + (A(x) - A(y))} (1 + a(y)) f(y)\, dy \\ &= \sum_{k=0}^\infty \frac{i^k}{k!} C_k(a, (1+a)f)(x), \end{aligned} \tag{6.1}$$

where the C_k are the one-dimensional Calderón commutators. Thus the Hilbert transform for the Lipschitz curve $\Gamma = \{x + iA(x) \colon x \in R\}$ may be viewed as the extension of this operation to purely imaginary functions.

The analogous statement is not true with respect to the Hilbert transform for Lipschitz hypersurfaces in $A_n(C)$: it does not arise in a natural way as an extension of the conjugation of $\mathscr{R}$ by an analogous change of variable. However, as we shall see, this conjugation in higher dimensions *is* a locally analytic function, in the sense of Theorem 5.7; and moreover, the Fréchet differentials of this operator-valued function are commutators of the type defined in Theorem 5.6.

Let $A = (A^1, \dots, A^n)$: $R^n \to R^n$ have bounded Jacobian $\alpha = J_A$; that is to say,

$$\|\alpha\|_\infty \equiv \sup_{j,k} \left\| \frac{\partial A^j}{\partial x_k} \right\|_\infty < \infty. \tag{6.2}$$

Let h: $R^n \to R^n$ be the bi-Lipschitz function given by $h(x) = x + A(x)$, and define $U_h f = f \circ h$. There is an open ball B centered at the origin in $L^\infty(R^n, M_n(R))$ such that, for all $\alpha \in B$, the function h is invertible (and hence U_h^{-1} exists).

Now let $\mathscr{R} = (R_1, \dots, R_n)$ denote, as before, the vector operator giving the Riesz transforms in R^n. If $f \in L^2(R^n, C)$, then for $x \in R^n$,

$$\mathscr{R}f(x) = \frac{-2}{\omega_{n+1}} \text{p.v.} \int_{R^{n+1}} \frac{x-y}{|x-y|^{n+1}} f(y)\, dy, \tag{6.3}$$

where

$$\omega_{n+1} = \frac{\pi^{(n+1)/2}}{2\Gamma((n+1)/2)} \tag{6.4}$$

is the surface area of the unit sphere in R^{n+1}. Consequently, for all $\alpha \in B$,

$$U_h \mathscr{R} U_h^{-1} f(x) = \frac{-2}{\omega_{n+1}} \text{p.v.} \int \frac{x - y + (A(x) - A(y))}{|x - y + (A(x) - A(y))|^{n+1}} g(y)\, dy, \tag{6.5}$$

where $g(y) = |J_h(y)| f(y)$, J_h is the Jacobian matrix of h and $|J_h|$ is its determinant. An elementary but extremely tedious calculation shows that

$$T(\alpha) f = U_h \mathscr{R} U_h^{-1} f = \sum_{j=0}^{\infty} K_j(\alpha) g, \tag{6.6}$$

where $K_j(\alpha)$ is an operator homogeneous of degree j in α, given by integration against the kernel

$$\frac{-2}{\omega_{n+1}} \frac{x-y}{|x-y|^{n+1}} \tag{6.7}$$

for $j = 0$, and

$$\begin{aligned} &\sum_{k=\nu}^{j} \gamma(n, j, k) G_{2k-j}(x, y) H_{2j-2k}(x, y) |x-y|^{-n-1-2k} (x-y) \\ &\quad + \sum_{k=\mu}^{j} \gamma(n, j-1, k-1) G_{2k-j}(x, y) H_{2j-2k}(x, y) |x-y|^{-n-1-2k} \end{aligned} \tag{6.8}$$

for $j \geq 1$, where $\nu = [(j+1)/2]$, $\mu = [j/2] + 1$, and

$$G_{2k-j}(x, y) = [(x-y)\cdot(A(x) - A(y))]^{2k-j}, \tag{6.9}$$

$$H_{2j-2k}(x, y) = |A(x) - A(y)|^{2j-2k}, \tag{6.10}$$

$$\gamma(n, j, k) = \frac{(-1)^{k+1} 2^{2k-j} \Gamma(k + (n+1)/2)}{\pi^{(n+1)/2} \Gamma(2k - j + 1) \Gamma(j - k + 1)}. \tag{6.11}$$

Further calculation (cf. [10, Chapter 3]) shows that

$$\gamma(n, j, k) G_{2k-j}(x, y) H_{2j-2k}(x, y) |x - y|^{-n-1-2k} (x - y) \tag{6.12}$$

is the kernel of an operator which is, in turn, the sum of n^k operators of the form

$$\lambda(n, j, k) T_{2k}(B^1, \ldots, B^{2k}), \tag{6.13}$$

where

$$\lambda(n, j, k) = \frac{-2^{2k-j-1} k!}{(2k-j)!(j-k)!(2k)!}; \tag{6.14}$$

T_{2k} is the vector operator $(T_{1,2k}, T_{2,2k}, \ldots, T_{n,2k})$, where $T_{l,2k}$ is defined for $1 \leq l \leq n$ by (5.52); and $B^1, \ldots, B^j \in \{A^1, \ldots, A^n\}$, $B^{j+1}, \ldots, B^{2k} \in \{X^1, \ldots, X^n\}$, the coordinate projections on R^n.

Similarly, we see that

$$\gamma(n, j-1, k-1) G_{2k-j}(x, y) H_{2j-2k}(x, y) |x - y|^{-n-1-2k} \tag{6.15}$$

is the kernel of an operator which is, in turn, the sum of n^k operators of the form

$$\lambda_0(n, j, k) T_{l,2k+1}(B^1, \ldots, B^{2k+1}), \tag{6.16}$$

where

$$\lambda_0(n, j, k) = \frac{-2^{2k-j} k!}{(2k-j-1)!(j-k)!(2k+1)!} \frac{1}{(n+2k-1)}; \tag{6.17}$$

$T_{l,2k+1}$ is defined by (5.52) with $1 \leq l \leq n$; and $B^1, \ldots, B^j \in \{A^1, \ldots, A^n\}$, $B^{j+1}, \ldots, B^{2k+1} \in \{X^1, \ldots, X^n\}$. Consequently, for $j \geq 1$ we have, by Theorem 5.6,

$$\begin{aligned} \|K_j(\alpha)\|_{\text{op}} \leq & \sum_{k=\nu}^{j} |\lambda(n, j, k)| n^{k+1} (2k)! (C(n))^{2k} \|\alpha\|_\infty^j \\ & + \sum_{k=\mu}^{j} |\lambda_0(n, j, k)| n^k (2k+1)! (C(n))^{2k+1} \|\alpha\|_\infty^j, \end{aligned} \tag{6.18}$$

where the operator norm is from $L^2(R^n, C)$ to $L_0^2(R^n, C)$. Combining (6.14), (6.17) and (6.18), we see that

$$\|K_j(\alpha)\|_{\text{op}} \leq k(n)^j \|\alpha\|_\infty^j, \tag{6.19}$$

where $k(n)$ is a purely dimensional constant. In view of the fact that the definition of $K_j(\alpha)$ and the estimate (6.19) are also valid for α with complex entries, we have

THEOREM 6.1. *Let $A = (A^1, \ldots, A^n)$: $R^n \to R^n$ and suppose that α, the Jacobian of A, is an element of $L^\infty(R^n, M_n(R))$. Set $h(x) = x + A(x)$ and define $U_h f = f \circ h$; let*

B be a neighborhood of 0 in $L^\infty(R^n, M_n(R))$ such that h is invertible for every $\alpha \in B$. For all $\alpha \in B$, define $T(\alpha) = U_h \mathscr{R} U_h^{-1}$. Then

(a) *the Fréchet differentials of T are sums of n-dimensional commutators of the type defined in Theorem 5.6; and*

(b) *T has a natural extension to functions α having complex entries, which is a complex-analytic operator-valued function in a neighborhood of the origin in $L^\infty(R^n, M_n(C))$.*

This theorem may be extended to more general Calderón-Zygmund operators, as follows.

THEOREM 6.2. *Let K be a bounded operator on $L^2(R^n, C)$ which commutes with translations and dilations, and suppose that its symbol, Ω, is a real-analytic function on Σ_{n-1}, the unit sphere in R^n. Then the mapping $\alpha \to U_h K U_h^{-1}$, defined for $\alpha \in B \subseteq L^\infty(R^n, M_n(R))$, is the restriction of a mapping $\alpha \to K_\alpha$ which is complex-analytic in a neighborhood of the origin in $L^\infty(R^n, M_n(C))$.*

PROOF. For ease of notation we shall adopt the following conventions. We shall let V denote the neighborhood of the origin in $L^\infty(R^n, M_n(C))$ on which the mapping T of Theorem 6.1 is complex-analytic. If $P(\mathscr{R}) = P(R_1, \ldots, R_n)$ is any polynomial in the Riesz transforms, and if $\alpha \in V$, we shall write $P(\mathscr{R}_\alpha) = (P(\mathscr{R}))_\alpha$ to denote the operator $P(T(\alpha))$. Clearly, $\alpha \to P(\mathscr{R}_\alpha)$ is a well-defined complex-analytic mapping in V.

Let $L^\infty(\Sigma_{n-1})$ and $L^2(\Sigma_{n-1})$ denote the spaces of essentially bounded and square-integrable complex-valued functions, respectively, with respect to normalized surface measure on the unit sphere in R^n. The space $L^2(\Sigma_{n-1})$ is equal to the orthogonal direct sum

$$\bigoplus_{k=0}^{\infty} H_k, \tag{6.20}$$

where H_k is the space of surface spherical harmonics of degree k on Σ_{n-1}. Let d_k denote the dimension of H_k; then $d_0 = 1$, $d_1 = n$, and in general, there is a constant $C > 0$ such that $d_k \leq C k^{n-2}$ for every k (see [**11**, Chapter 4]). If we let

$$\mathscr{S}_k = \left\{ Y_1^{(k)}, \ldots, Y_{d_k}^{(k)} \right\} \tag{6.21}$$

be an orthonormal basis for H_k, then the union of the $\mathscr{S}_k$ is an orthonormal basis for $L^2(\Sigma_{n-1})$. With respect to this orthonormal basis, Ω has the Fourier expansion

$$\sum_{k=0}^{\infty} P_k, \tag{6.22}$$

where $P_k \in H_k$ is given by

$$P_k = \sum_{j=1}^{d_k} \left(\Omega, Y_j^{(k)}\right) Y_j^{(k)} \tag{6.23}$$

and

$$\left(\Omega, Y_j^{(k)}\right) = \int_{\Sigma_{n-1}} \Omega(\xi') \overline{Y_j^{(k)}(\xi')}\, d\xi'. \tag{6.24}$$

Since Ω is a real-analytic function on Σ_{n-1}, it follows, by a theorem of Morrey and Nirenberg (see [9]) that Ω can be extended to a function $\tilde{\Omega}$ which is harmonic in a neighborhood of the unit ball in R^n. In particular, there is a constant $r_0 > 1$ such that, if we set $|\xi| = r$ and $\xi' = \xi r^{-1}$, then for $r < r_0$,

(6.25) $$\tilde{\Omega}(\xi) = \sum_{k=0}^{\infty} r^k P_k(\xi').$$

This series converges absolutely for $r < r_0$; moreover, there are positive constants C_0 and M_0, with $C_0 < r_0^{-1}$, such that

(6.26) $$\left|\left(\Omega, Y_j^{(k)}\right)\right| \leq M_0 C_0^k.$$

Now let us define, for $\alpha \in V$, the operator K_α by setting

(6.27) $$K_\alpha = \sum_{k=0}^{\infty} P_k(\mathscr{R}_\alpha) = \sum_{k=0}^{\infty} \sum_{j=1}^{d_k} \left(\Omega, Y_j^{(k)}\right) Y_j^{(k)}(\mathscr{R}_\alpha).$$

We claim that the mapping $\alpha \to K_\alpha$ is complex-analytic in a neighborhood of the origin in V, and that its restriction to functions having only real entries is precisely the conjugation of K by the change of variable U_h.

It suffices to show that the series $\sum_{k=0}^{\infty} P_k(\mathscr{R}_\alpha)$ converges absolutely and uniformly to K_α for all α in a neighborhood of the origin in V. To this end, we will obtain uniform estimates for

(6.28) $$\sup_{1 \leq j \leq d_k} \left\| Y_j^{(k)}(\mathscr{R}_\alpha) \right\|_{\text{op}}$$

(where the operator norm is from $L^2(R^n, C)$ to itself) for all α in some neighborhood of the origin in V.

We shall make use of the following observations. If ρ is a rotation of R^n, then the operator U_ρ, defined for $f \in L^2(R^n)$ by $U_\rho f = f \circ \rho$, is an isometry of $L^2(R^n)$. If we define the operator $\rho\mathscr{R}$ to be the vector operator given by the symbol $i\rho\xi'$, then the jth entry of $\rho\mathscr{R}$, denoted $(\rho\mathscr{R})_j$ is equal to $U_\rho R_j U_\rho^{-1}$. Now, if α has only real entries, we have

(6.29) $$\left[(\rho\mathscr{R})_j\right]_\alpha = U_\rho U_k R_j U_k^{-1} U_\rho^{-1},$$

where U_k is the operator of composition with $k = \rho \circ h \circ p^{-1}$. Now $k(x) = x + A_0(x)$, where $A_0 = \rho \circ A \circ \rho^{-1}$; the Jacobian matrix α_0 of A_0 is equal to $\rho \cdot (\alpha \circ \rho^{-1}) \cdot \rho^{-1}$. Now, α_0 has the same L^∞ norm as α; moreover,

(6.30) $$\left[(\rho\mathscr{R})_j\right]_\alpha = U_\rho \left[(R_j)_{\alpha_0}\right] U_\rho^{-1}.$$

Furthermore, it is not difficult to see that (6.30) continues to hold if α has complex entries. Since U_ρ is an isometry, we have

(6.31) $$\left\|\left[(\rho\mathscr{R})_j \pm i(\rho\mathscr{R})_k\right]_\alpha\right\|_{\text{op}} = \left\|\left(R_j \pm iR_k\right)_{\alpha_0}\right\|_{\text{op}}$$

for all $\alpha \in V$, all $j, k \in \{1, \ldots, n\}$, and all rotations ρ.

In view of the fact that the mapping $\alpha \to (R_j)_\alpha$ is complex-analytic and therefore continuous in V, we see that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\alpha \in V$ with $\|\alpha\|_\infty < \delta$, then

$$\|[(\rho\mathscr{R})_j \pm i(\rho\mathscr{R})_k]_\alpha\|_{\mathrm{op}} < 1 + \varepsilon \tag{6.32}$$

for all $j,k \in \{1,\ldots,n\}$ and for every rotation ρ.

Now, let $SO(n)$ denote the group of proper rotations on R^n, and let $d\rho$ be the element of normalized Haar measure on $SO(n)$. Let η denote the north pole on Σ_{n-1}; i.e., $\eta = (1,0,\ldots,0)$. We let $Z_\eta^{(k)}$ denote the zonal harmonic of degree k with pole η (see [**11**, Chapter 4], for properties of $Z_\eta^{(k)}$). It is not difficult to show that, for $1 \le j \le d_k$ and $\xi' \in \Sigma_{n-1}$,

$$Y_j^{(k)}(\xi') = \int_{SO(n)} Y_j^{(k)}(\rho\eta)\,\overline{Z_\eta^{(k)}(\rho^{-1}\xi')}\,d\rho. \tag{6.33}$$

Consequently

$$Y_j^{(k)}(\mathscr{R}_\alpha) = \int_{SO(n)} Y_j^{(k)}(\rho\eta)\,\overline{Z_\eta^{(k)}((\rho^{-1}\mathscr{R})_\alpha)}\,d\rho. \tag{6.34}$$

Now suppose that $x' = (x_1', x_2', \ldots, x_n') = (x_1', \tilde{x}) \in \Sigma_{n-1}$. We claim that

$$Z_\eta^{(k)}(x') = d_k \int_{\Sigma_{n-2}} (x_1' + i\tilde{x}\cdot\tilde{y})^k\,d\tilde{y}, \tag{6.35}$$

where $d\tilde{y}$ is the element of normalized surface measure on Σ_{n-2}. This follows from the fact that $Z_\eta^{(k)}$ is the unique element of H_k which takes on the value d_k at η and is invariant under rotations that leave η fixed (see [**11**, Chapter 4]). If $d\tilde{\rho}$ is the element of normalized Haar measure on $SO(n-1)$ and $\tilde{\eta} = (1,0,\ldots,0) \in \Sigma_{n-2}$, then we have

$$\begin{aligned} Z_\eta^{(k)}(x') &= d_k \int_{SO(n-1)} (x_1' + i\tilde{x}\cdot\tilde{\rho}\tilde{\eta})^k\,d\tilde{\rho} \\ &= d_k \int_{SO(n-1)} (x_1' + i(\tilde{\rho}^{-1}\tilde{x})_1)^k\,d\tilde{\rho}, \end{aligned} \tag{6.36}$$

where $(\tilde{\rho}^{-1}\tilde{x})_1$ denotes the first component of $\tilde{\rho}^{-1}\tilde{x}$. Consequently, if $\rho \in SO(n)$ and $\alpha \in V$, then

$$\overline{Z_\eta^{(k)}((\rho^{-1}\mathscr{R})_\alpha)} = d_k \int_{SO(n-1)} \left\{[(\rho\tilde{\rho})^{-1}\mathscr{R}]_1 - i[(\rho\tilde{\rho})^{-1}\mathscr{R}]_2\right\}_\alpha^k d\tilde{\rho}, \tag{6.37}$$

where we consider $\tilde{\rho}$ both as an element of $SO(n-1)$ and as an element of $SO(n)$ which leaves η fixed.

If we choose $\varepsilon > 0$ so that $(1+\varepsilon) < C_0^{-1}$ (where C_0 is the constant occurring in (6.26)), then we can find $\delta > 0$ so that if $\alpha \in V$ with $\|\alpha\|_\infty < \delta$,

$$\left\|\overline{Z_\eta^{(k)}((\rho^{-1}\mathscr{R})_\alpha)}\right\|_{\mathrm{op}} \le d_k(1+\varepsilon)^k \tag{6.38}$$

by (6.32) and (6.37). Thus, by (6.34),

$$\left\| Y_j^{(k)}(\mathscr{R}_\alpha) \right\|_{\text{op}} \leq d_k(1 + \varepsilon)^k. \tag{6.39}$$

Thus, by (6.26), (6.27), and (6.39), it follows that the series (6.27) converges absolutely and uniformly to K_α for all $\alpha \in V$ satisfying $\|\alpha\|_\infty < \delta$. Thus the proof is complete by the remark preceding (6.28). Q.E.D.

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